A THURSTONE-TYPE MODEL FOR PAIRED COMPARISONS WITH UNEQUAL NUMBERS OF REPETITIONS—I

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(Received : November, 1976)

SUMMARY

A modification of Thurstone Model for analysis of data from paired comparison experiments has been given by Mosteller. In this paper angular transformation is used to generalise Mosteller's model in order to make the preference proportions independent and incidentally ensure homoscedasticity of variances and correlations and additivity of scale in the subjective continuum for the stimuli.

1. INTRODUCTION

The method of paired comparisons is a technique of ranking stimuli by the response they produce in a subject by offering the stimuli in all possible pairs, the number of repetitions on each pair being equal or unequal. When ties and order effects are ignored the judge is to express his preference by giving a score 1 to the preferred object of the pair and a score zero to the non-preferred object. Thurstone (1927) developed a model for analysis of data from such experiments under the following assumptions:—

(*i*) There is a set of stimuli which can be located in a subjective continuum.

(*ii*) Each stimulus when presented to an individual gives rise to a sensation in the individual.

(*iii*) The distribution of sensations for a particular stimulus over repetitions is normal.

(*iv*) The stimuli are presented in pairs to an individual thus giving rise to a sensation for each stimulus.

(*v*) Assume equal standard deviations for each stimulus and zero correlations between pairs of stimuli. Mosteller (1951 a) relaxed the condition of zero correlations in (*v*) to one of equal correlations with no change in method. In (1951 b) he discussed the effect of an aberrant standard deviation. In (1951 c) he used an angular transformation to develop a validity test for his model. Glen and David (1960) used an inverse sine transformation to solve a problem of correlated data due to ties in paired comparison experiments. In the present paper we use angular transformation to the basic preference data and develop two rating scales along with their dispersion matrices for equal as well as unequal numbers of repetitions in paired comparisons. The logic of the transformation is discussed in section 2. Specialisations of the first scale are given for two incomplete patterns of paired comparisons namely, standard comparison pairs and symmetrical pairs by Sadasivan (1973, 1974, 1977). Further the Pitman efficiency of S.C.P. is derived.

2. The Model

Let X_i, X_j be single sensations evoked in a judge by the i-th and j-th stimuli. Assume X_i, X_j to be normally distributed with

$$E(X_i)=S_i;$$

$$\text{Var}\ (X_i)=\sigma^2 \qquad (i=1, 2, .t);$$

correlation between X_i, $X_j=c$, a constant. With error-free data we can order the stimuli as follows :

Let P_{ij} be the proportion of times $X_i>X_j$. Then using the same methods as in (3)

$$P_{ij}=F(S_i-S_j)\ .$$

where

$$F(X)=\frac{1}{\sqrt{2\pi}}\int_{-X}^{\infty} e^{\frac{-1}{2}y^2}\,dy... \qquad ...(2{\cdot}2)$$

Given any P_{ij} we can solve for $-(S_i-S_j)$ by using the normal probability tables. We get $t(t-1)/2$ equations involving t unknowns. Arbitrarily set $S_1=0$ and solve for S_i $(i=2,..\ t)$.

In the case of experimental data the estimate of $P_{ij}=p_{ij}=$the proportion of preference for i, from n_{ij} trials of the pair (i, j). Then

$$p_{ij}=F(D'_{ij})$$

where

$$D'_{ij}=(S'_i - S'_j)$$

$$=\text{estimate of } (S_i-S_j).$$

Under Mosteller's model, the distribution of the difference X_i-X_j is normal with mean S_i-S_j and unit variance. Further,

$$F(-x)=1-F(x). \quad \text{Hence}$$

$$S'_i - S'_j = F^{-1}(p_{ij}).$$

Mosteller (3) gets the treatment ratings directly from the p_{ij} values using summation and least square techniques. For his technique to be valid (*i*) X_i ($i=1, 2,...t$) should be normally distributed with mean S_i ($i=1, 2, ..t$) and uniform variance σ^2, (2) X_i, X_j ($i \neq j$) must be equally correlated and (3) the scale should be additive. If any of these conditions fail, the ratings will not reflect the true situation. It may be noted that the assumption of normality is only a computational device for building up a rating scale. Moreover, the variability of X_i ($i=1, 2,...t$) will not always be uniform as postulated. An extraneous factor as taste fatigue in sensory discrimination will affect the variability. Unequal correlations among the responses will also affect the ratings. The condition that the scale should be additive is often broken in paired comparisons. Further, it may happen that different judges have entered into the preference testing giving rise to different correlations among pairs of stimuli. The transformation takes care of these possible disturbances. Moreover, Mosteller's technique does not give an estimate of variance of ratings. Hence we modify the assumptions under the model as (1) the responses are additive along the transformed scale. (2) The variability of X_i is uniform along the transformed scale. (3) Pairs of responses are equally correlated under the new scale. (4) The distribution of sensations for a particular stimulus becomes normal under the new scale.

Now use the inverse sine transformation,

$$\theta' = \text{arc sin } \sqrt{p'}$$

where p' is the observed proportion from a binomial sample of size n from a population with true proportion of success p. θ' is approximately normally distributed with nearly independent variance $\sigma^2_{\theta'} = 821/n$ or $1/4n$ depending on whether θ' is measured in degrees or radians. Proceeding as before we get the same expression (2·2) under the modified assumptions. Replace (2·2) by

$$F(X) = \tfrac{1}{2} \int_{-X}^{\pi/2} \cos y \, dy = \tfrac{1}{2} (1 + \sin x) \left(-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \right)$$

where X represents an angle in radian measure. This change is possible under the modified assumptions. Then

$$S_i' - S_j' = \sin^{-1} (2p_{ij} - 1) = D_{ij}$$

which can be obtained from tables of angular transformations. Also,

$$-S_i' + S_j' = \sin^{-1}(2p_{ji}-1) = -\sin^{-1}(2p_{ij}-1).$$

For large samples,

$$\text{Var}\ \sin^{-1}(2p_{ij}-1) = \text{Var}\ \sin^{-1}(2p_{ji}-1) = \text{Var}(S_i' - S_j') = \frac{1}{n_{ij}}$$

Then the error sum of squares

$$E = \sum_{i<j} (S_i - S_j - D_{ij})^2 \qquad \ldots(2\cdot3)$$

Minimise (2·3) with respect to S_i $(i=1, 2, \ldots t)$. Set $S_1=0$.

Then
$$\sum_{j=2}^{t} S_j = \sum_{j=2}^{t} D_{ij} \qquad \ldots(2\cdot4)$$

$$(t-1)S_i - \sum_j{}' S_j = \sum_j D_{ij} \qquad (i=2, 3, \ldots t) \qquad \ldots(2\cdot5)$$

where $\sum_j{}'$ is summation $j \neq i$. The system (2·5) is linearly independent and hence permits unique solution. Putting (2·5) in matrix form and solving, the estimate of $S = S^* = (X'X)^{-1}D$

where
$$(X'X)^{-1} = \frac{1}{t}\begin{bmatrix} 2 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 1 & \ldots & 1 \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ 1 & 1 & 1 & \ldots & 2 \end{bmatrix}, \quad S' = (S_2, S_3 . St) \text{ and}$$

$$D' = [\sum_j{}' D_{2i}, \ldots \sum_j{}' D_{ij}]$$

The dispersion matrix of S^* is

$$\sum_{S^*} = (X'X)^{-1} \sum_{\mathbf{D}} (X'X)^{-1} \qquad \ldots(2\cdot7)$$

where $\sum_{\mathbf{D}}$ is the variance-covariance matrix of D.

when
$$n_{ij} = n, \quad \sum_{S^*} = (X'X)^{-1}/n \qquad \ldots(2\ 8)$$

A different method of solution is by using Rao (1974) chapter 4. The unreduced normal equations after transformation are $(X'X)S=D_1$ where D_1 is a colum vector of t elements. Find the g-inverse C of $X'X$. Then $S^*=CD_1$ and dispersion of D_1 *i.e.* $D(D_1)=\sigma^2\,(X'X)$ where σ^2 is the common-factor of the elements of the dispersion matrix for a constant number n of repetitions on each pair. Hence dispersion of $S^*=\frac{C}{n}$. When $n=n_{ij}$, the observational equations can be set out as

$$E(D)=XS; \quad D(D)=\Sigma\,; \quad |\Sigma|\neq 0. \quad \text{Put} \quad Z=\Sigma^{-\frac{1}{2}}D$$

Then $E(Z)=\Sigma^{-\frac{1}{2}}XS=US$ (say); $D(Z)=I$.

The normal equations are $U'US=U'Z=D_2$. Find the g-inverse C_1 of $U'U$. Then $S^*=C_1D_2$ and $\sum\limits_{S^*}=C_1$.

3. An Alternative Model

Using the classical Gauss-Markoff set-up our problem can be formulated as follows. The observational equations after angular transformation can be put as

$$D=XS+\epsilon$$

where X is a $\frac{t(t-1)}{2}\ X\ t$ matrix,

$$D'=\left(D_1,\ D_2,.\ \frac{Dt(t-1)}{2}\right),\ S'=(S_1,\ S_2,\ldots S_t).$$

and $\epsilon=$the error vector.

Then $E(\epsilon)=0$;

$D(\epsilon)=\sigma^2I$

is the dispersion matrix which means the error-variables are uncorrelated after transformation. Under the original scale ϵ_i, the error corresponding to the ith stimulus also depends on the other stimuli as is evident from the normal equations and is hence correlated. The transformation takes care of this correlation. Thus

$$E(D)=XS\,; \quad D(D)=\frac{1}{n}I$$

where the number of repetitions on a pair $(i, j)=n$. Then the normal equations are

$$X'XS=X'D \qquad \ldots(3{\cdot}1)$$

(3·1) does not admit a solution since $R(X)$, the rank of $X=(t-1)<t$, the numher of parameters to be estimated. In order to increase $R(X)$ to t, introduce the constraint $S_1+S_2+\ldots+S_t=0$. This constraint is valid since we are interested only in the relative

positions of the stimuli. This provides a matrix H of size $(1\times t)$ such that $R(X':H')=t$, that is we add one row to meet the deficiency in rank. Hence, the true inverse $C=(X'X+H'H)^{-1}$=a g-inverse of $X'X$. Then a unique solution of the ratings is given by

$$S^*=CX'D=CQ \text{ with dispersion}$$

$$D(S^*)=\sigma^2C=1/n\,(X'X+H'H)^{-1} \quad \text{where} \quad Q=X'D.$$

For any linear functions $P'S^*$, $R'S^*$, the variances and covariances are

$$V(P'S^*)=\sigma^2P'CP \text{ and}$$

$$\text{Cov}\,(P'S^*, R'S^*)=\sigma^2P'CR$$

where P', R' are row vectors of t elements each.

When $n=n_{ij}$ for pair (i, j) use the transformation

$$Z=\Sigma^{-\frac{1}{2}} D \text{ to the observational equations } D=XS;$$

$$D(D)=\Sigma,\ |\Sigma|\neq 0$$

so that $E(Z)=US$

where $U=\Sigma^{-\frac{1}{2}} X$

and $D(Z)=I.$

The normal equations become

$U'US=U'Z$ and $C=(U'U+H'H)^{-1}$ where H is defined as above. Thus the rating vector and the corresponding dispersion matrix are given by $S^*=CU'S$; $D(S^*)=C$. The variances and covariances of linear functions of rating vectors are obtained as $V(P'S^*)=P'CP$; cov $(P'S^*, R'S^*)=P'CR$. A model for standard comparison pairs is obtained by putting $n_{ij}=n$ for pairs compared and $n_{ij}=0$ for others. Then (2·6), (2·8) and (2·7) reduce respectively to

(*i*) $S^*=-D$;

(*ii*) $\Sigma_{S^*}=\frac{1}{n}I$;

(*iii*) $\Sigma_{S^*}=\left(\frac{1}{n_{ij}}\right)[(t-1)\times(t-1)]$

For t stimuli, $S_1, S_2 \ldots S_t$, a set of symmetrical pairs is (S_1, S_2) $(S_2, S_3)\ldots(S_t, S_1)$. In the model in section 2 substitute $n_{i,\,i+1}=n$ for $i=1, 2, \ldots t$ with the convention that $(i+1)$ mod t must be taken for $i=t$ and $n_{ij}=0$ for $j\neq i-1$. Then the solutions (2·6), (2·8) and (2·7) for symmetrical pairs are respectively

(*iv*) $S^*=(X'X)^{-1}D$ with $D=\begin{pmatrix} -D_{12}+D_{23} \\ -D_{23}+D_{34} \\ \cdots\cdots\cdots \\ -D_{t-1,t}+D_{t1} \end{pmatrix}$

(v) $$\sum_{S^*} = \frac{(X'X)^{-1}}{n}$$

(vi) $$\sum_{S^*} = (X'X)^{-1} \sum_{D} (X'X)^{-1};$$

4. Tests

As a test of validity of the model one can use a chi-square in the usual way as given by Sadasivan *et al.* (1974). An exact overall test for stimuli can be made by using the theorem of Rao (1974) which states that the distribution of a quadratic form $Z' \overset{-1}{\sum} Z$ is a non-central χ^2 where Z is a vector of normal variables with mean μ and dispersion matrix, $\sum$.

5. Efficiency of S. C. P. and Conclusions

The test statistic for complete pairs with unequal number of repetitions *viz*

$$S^{*\prime} \sum_{S^*}^{-1} S^* \text{ is distributed as a non-central}$$

$$\chi^2\,(t-1,\, S' \sum_{S}^{-1} S) \qquad \text{and that for } S.C.P. \text{ viz}$$

$$S_s^{*} \sum_{S_s^*}^{-1} S_s^{*} \text{ is distributed as a}$$

$$\chi^2(t-1, S_s' \sum_{S_t}^{-1} S_s)$$

where S and S_s are the respective rating vectors. Now we derive the Pitman efficiency of *S.C.P.* using the method of Noether (1955) to test the hypothesis.

$$H_0 : Si=0 \; (i=2,\ldots t) \qquad \text{against the alternative}$$

$$H_1 : Si = \frac{\text{Sin}}{n} \; (i=2, 3,\ldots t)$$

where S_{in} are arbitrary, positive constants and n, the sample size. Using the approximations,

(*a*) $$\sum_{i=2}^{t} S_i = 0$$

for nearby alternatives to the null hypothesis for full pairs

(*b*) $-D_{1j}=S_j$ for *S.C.P.* the *ARE* of *S C.P.* to full pairs is $(t-1)/t$. It may be noted that for more rapidly converging alternatives this efficiency increases.

Thus the present paper generalises Mosteller's model for paired comparisons to unequal correlations among the pairs.

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